CSE311 Microwave Engineering

LEC (03) – Chapter (03)
Lec (03)
General Plane Wave Solution

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1.4 Maxwell's Equations

1.4.1 Maxwell's Equations in Point or Differential Form

- We have already obtained two of the Maxwell's equations for time-varying fields which are:

$$\nabla x \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ **Faraday's Law of induction** (1.15)

$$\nabla x \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$ **Ampere's Circuital Law** (1.16)

- The remaining two equations are unchanged from their non-time-varying from which are:

$$\nabla . \vec{D} = \rho_v$$ **Gauss's Law for Electric Field** (1.17)

$$\nabla . \vec{B} = 0$$ **Gauss's Law for Magnetic Field** (1.18)

- Equation (1.18) acknowledges the fact that "magnetic charges" or poles are not known to exist. Magnetic flux is always found in closed loops and never diverges from a point source.

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H} \qquad \vec{J} = \sigma \vec{E}$$

2.1 Introduction

- Similarly, $\varepsilon_o = 8.854 \times 10^{-12} = (1/36\pi)\times 10^{-9}$ F/m) is the permittivity of free-space and $\varepsilon_r > 1$ is the relative permittivity of the material.
- Loss tangent values will determine types of media, the loss tangent is defined as: $$\tan\delta = \sigma / \omega\varepsilon$$
- Loss tangent values will determine types of media:
  - tan δ small ($\sigma / \omega\varepsilon < 0.1$) – good dielectric
  - tan δ large ($\sigma / \omega\varepsilon > 10$) - good conductor
- Another factor that determined the characteristic of the media is operating frequency, f or the angular frequency $\omega = 2\pi f$. A medium can be regarded as a good conductor at low frequency might be a good dielectric at higher frequency.
- In this chapter, our major goals are to:

1. Use Maxwell's equations to derive the wave equations in case of the medium is free of charge, $\rho_v = 0$ and then solve the wave equations to determine the wave characteristics and parameters in the following media:

   a. General medium ($\sigma \neq 0$, $\mu = \mu_r \mu_o$ and $\varepsilon = \varepsilon_r \varepsilon_o$ and $\sigma \approx \omega\varepsilon$) lossy medium.

   b. Free space or air ($\sigma = 0$, $\mu = \mu_o$ and $\varepsilon = \varepsilon_o$).

   b. Lossless (perfect) dielectric or medium ($\sigma = 0$, $\mu = \mu_r \mu_o$ and $\varepsilon = \varepsilon_r \varepsilon_o$).

   c. Lossy (imperfect) dielectric or medium ($\sigma \neq 0$, $\mu = \mu_r \mu_o$, $\varepsilon = \varepsilon_r \varepsilon_o$ and $\sigma << \omega\varepsilon$).

   d. Good conductor medium ($\sigma \neq 0$, $\mu = \mu_r \mu_o$, $\varepsilon = \varepsilon_r \varepsilon_o$ and $\sigma >> \omega\varepsilon$).

2.2 General Plane Wave Equations

- The analytical treatment of the uniform plane wave is consider when the wave propagates in dielectric material of conductivity, σ, permittivity, $\varepsilon = \varepsilon_r\varepsilon_o$ and permeability, $\mu = \mu_r\mu_o$.
- The medium is assumed to be homogenous (having constant μ and ε with position) and isotropic (in which μ and ε are invariant with the field orientation). Assume the medium is free of charge, so $\rho_v = 0$, under these conditions, Maxwell's equations in point form may be written in terms of $\vec{E}$ and $\vec{H}$ only as:

$$\nabla x\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu\frac{\partial \vec{H}}{\partial t} \qquad (2.1)$$

$$\nabla x\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t} \qquad (2.2)$$

$$\nabla.\vec{E} = \frac{\rho_v}{\varepsilon} = 0 \qquad (2.3) \qquad\qquad \nabla.\vec{H} = 0 \qquad (2.4)$$

$\vec{E}$ is the electric field intensity, in volts per meter (V/m).
$\vec{H}$ His the magnetic field intensity, in amperes per meter (A/m).
$\vec{D}$ is the electric flux density, in coulombs per meter squared (C/m^2).
$\vec{B}$ is the magnetic flux density, in webers per meter squared (Wb/m^2).
$\vec{J}$ is the electric current density, in amperes per meter squared (A/m^2).
ρ_v is the electric volume charge density, in coulombs per meter cubed (C/m^3)

$$\frac{\partial \vec{E}}{\partial t} = j\omega\vec{E} \qquad\qquad \frac{\partial \vec{H}}{\partial t} = j\omega\vec{H}$$

2.2 General Plane Wave Equations (Continued)

- In charge free, linear isotropic, homogenous region, applying Eq.(2.9), the Maxwell's equations [(2.1) to (2.4)] in the Phasor form are:

$$\nabla x\vec{E}_s = -j\omega\mu\vec{H}_s \qquad (2.10)$$

$$\nabla x\vec{H}_s = \sigma\vec{E}_s + j\omega\varepsilon\vec{E}_s = (\sigma + j\omega\varepsilon)\vec{E}_s \qquad (2.11)$$

$$\nabla.\vec{E}_s = \frac{\rho_v}{\varepsilon} = 0 \qquad (2.12)$$

$$\nabla.\vec{H}_s = 0 \qquad (2.13)$$

- Equations from (2.10) to (2.13) constitute the two unknowns, $\vec{E}$ and $\vec{H}$. It required to solve only for one variable, either $\vec{E}$ or $\vec{H}$ as follows:

1. Take the Curl operator of both sides of (2.10), we have:

$$\nabla x\nabla x\vec{E}_s = -j\omega\mu_o\nabla x\vec{H}_s \qquad (2.14)$$

2. Apply the curl identity described in Eq.(A.5) we have: $\nabla^2\vec{E} = \nabla(\nabla.\vec{E}) - \nabla x\nabla x\vec{E}$

$$\nabla x\nabla x\vec{E}_s = \nabla(\nabla.\vec{E}_s) - \nabla^2\vec{E}_s \qquad (2.15a)$$

- From (2.12), we get:

$$\nabla x\nabla x\vec{E}_s = 0 - \nabla^2\vec{E}_s = -\nabla^2\vec{E}_s \qquad (2.15b)$$

2.2 General Plane Wave Equations (Continued)

- From (2.11) and (2.15) substitute (2.14), the resulting wave equation for $\vec{E}$ then becomes:

$$\nabla^2 \vec{E}_s + \omega^2 \mu\varepsilon (1 - j\frac{\sigma}{\omega\varepsilon})\vec{E}_s = 0 \quad (2.16a)$$

$$\nabla^2 \vec{E}_s - \gamma^2 \vec{E}_s = 0 \quad (2.16b)$$

- Where γ ***is the wave number or complex propagation constant*** of the medium. In this case γ is a function of the material properties, as described by μ, ε and σ. γ is a complex value as it is referred to as complex propagation constant for the medium defined by:

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} \quad (2.16c)$$

- Where α [Np/m] is called the attenuation constant or coefficient and β [rad/m] is the phase constant or coefficient.
- Eq. (2.16) is the wave equation, or Helmholtz equation, for $\vec{E}$.
- An identical wave equation for $\vec{H}$ can be derived in the same manner as:

$$\nabla^2 \vec{H}_s + \omega^2 \mu\varepsilon (1 - j\frac{\sigma}{\omega\varepsilon})\vec{H}_s = 0 \quad (2.17a)$$

$$\nabla^2 \vec{H}_s - \gamma^2 \vec{H}_s = 0 \quad (2.17b)$$

- Equations (2.16) and (2.17) are called the general plane wave equations or Helmholtz equations whose solutions describes the wave behavior and characteristics in a specific media as will be explained in the next sections.

2.3 Solution of Wave Equations in a General Medium

2.3.1 Classification of wave solution

- It is now possible to separate the solutions of the wave equation in a charge free region into three basic types of fields. These are:

1. Transverse Electro-Magnetic wave (TEM wave):

It is characterized by the condition that both the electric and magnetic field vectors lie in a plane perpendicular to the direction of propagation, i.e. having no components in the direction of propagation.

2. Transverse Electric wave (TE, or H-wave):

It is characterized by having an electric field which is entirely in a plane transverse to the (assumed) direction of propagation. Only the magnetic field $\vec{H}$ has a component in the direction of propagation and hence this wave type is also known as H-waves. For TE waves, it is possible to express all field components in terms of the axial magnetic field component.

3. Transverse Magnetic wave (TM, or E-wave)

It is characterized by having a magnetic field which in entirely in a plane transverse to the (assumed) direction of propagation. Only the electric field $\vec{E}$ has a component in the direction of propagation, and hence this wave type is also known as E-wave. For TM-wave, it is possible to express all field components in term of the axial electric field component.

2.3 Solution of Wave Equations in a General Medium (continued)

2.3.2 Solution of Wave Equation

- A basic plane wave solution to the above equations can be found by considering an electric field with only an $\hat{a}_x$ component and uniform (no variation) in the x and y, then we get:

$$\vec{E}_s = E_{xs}(z)\hat{a}_x \quad then \quad \frac{\partial E_{xs}(z)}{\partial x} = 0 \text{ and } \frac{\partial E_{xs}(z)}{\partial y} = 0 \tag{2.18}$$

- The Laplacian operator, ∇^2 is defined in Eq. (A.6) as: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$
- The Helmholtz or wave equation for $\vec{E}$ (2.16) is reduced to:

$$\frac{\partial^2 \vec{E}_{xs}(z)}{\partial z^2} - \gamma^2 \vec{E}_{xs}(z) = 0 \tag{2.19}$$

- The solution to Eq.(2.19) will be forward and backward propagating waves having the phasor solutions in the form:

$$E_{xs}(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z} = E^+ e^{-\alpha z} e^{-j\beta z} + E^- e^{\alpha z} e^{j\beta z} \tag{2.20}$$

- Where E^+ and E^- are arbitrary constants.
- The positive traveling wave in phasor form after substituting for γ is:

$$E_{xs}(z) = E^+ e^{-\gamma z} = E^+ e^{-\alpha z} e^{-j\beta z} \tag{2.21}$$

- Multiplying (2.21) by $e^{j\omega}$ and taking the real part yields a form of the field in time domain that can be visualized as:

$$E_x(z,t) = E^+ e^{-\alpha z} \cos(\omega t - \beta z) \tag{2.22}$$

2.3 Solution of Wave Equations in a General Medium (continued)

2.3.2 Solution of Wave Equation

- The above solution is for time harmonic case at frequency ω. In time domain, this result is written as:

$$E_x(z,t) = \mathrm{Re}[E_{xs}(z)e^{j\omega t}] = \mathrm{Re}\left[\left(E^+ e^{-\alpha z} e^{-j\beta z} + E^- e^{\alpha z} e^{j\beta z}\right) e^{j\omega t}\right]$$

$$E_x(z,t) = E^+ e^{-\alpha z}\cos(\omega t - \beta z) + E^- e^{\alpha z}\cos(\omega t + \beta z) \qquad (2.23)$$

- From Eqs. (2.22 and 23), it is seen that the $\vec{E}$ wave that travels in a conducting medium, its amplitude is attenuated by the factor e^{-az}, as shown in Fig. 2.1.
- Similarly, the Helmholtz or wave equation for $\vec{H}$ (2.17) can be solved to have:

$$H_{ys}(z) = H^+ e^{-\gamma z} + H^- e^{\gamma z} = H^+ e^{-\alpha z} e^{-j\beta z} + H^- e^{\alpha z} e^{j\beta z} \qquad (2.24)$$

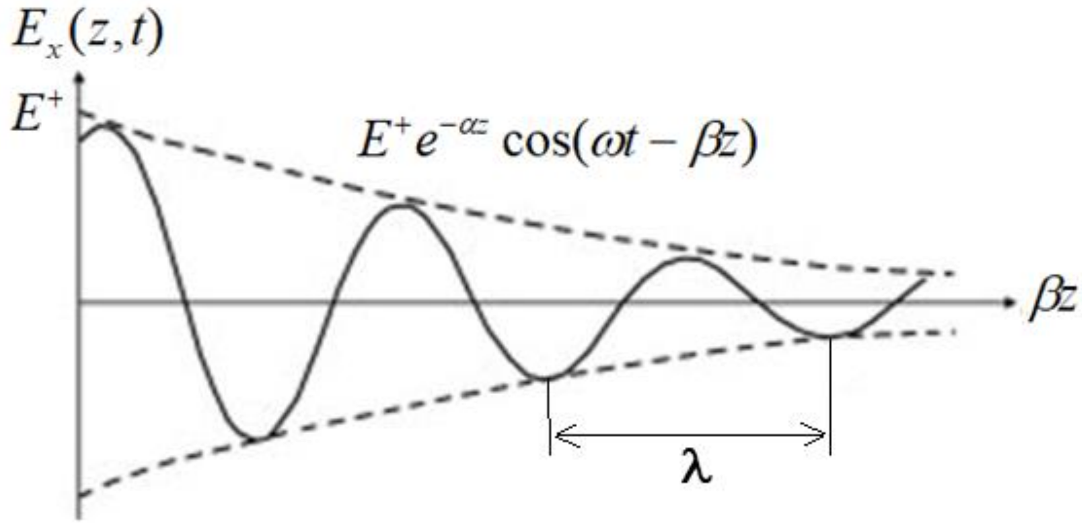


Fig. 2.1 The electric field wave travels in a conducting medium.

2.3 Solution of Wave Equations in a General Medium (continued)

2.3.3 Basic plane wave parameters

1. Phase velocity, v_p

- Consider the positive traveling wave described by Eq. (2.22) as:

$$E_x(z,t) = E^+ e^{-\alpha z} \cos(\omega t - \beta z)$$

- To maintain a fixed point on the wave, the phase is constant, therefore:

$$\text{phase} = (\omega t - \beta z) = \text{constant} \qquad (2.25)$$

- The velocity of the wave is called the phase velocity, because it is the velocity at which a fixed phase point on the wave travel and is given by dz/dt:

$$\frac{d}{dt}(\omega t - \beta z) = 0 \rightarrow \omega - \beta \frac{dz}{dt} = 0$$

- So, the phase velocity, v_p is given by:[rad/m]

$$\omega \ in \ [rad/s] \ \ and \ \ \beta \ in \ [rad/m]$$

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} \ \ [m/s] \qquad (2.26)$$

2.3 Solution of Wave Equations in a General Medium (continued)

2.3.3 Basic plane wave parameters (continued)

2. The Wavelength, λ

$$\beta \ in \ [rad/m]$$

- The wavelength λ, is defined as the distance between two successive maximum (or minima, or any other reference points) on the wave, at a fixed instant of time, thus: $[(\omega t - \beta z) - (\omega t - \beta (z + \lambda))] = 2\pi \rightarrow \beta \lambda = 2\pi$

So,

$$\lambda = \frac{2\pi}{\beta} \ [m] \quad and \ v_p = \frac{\omega}{\beta} \quad \Rightarrow \lambda = \frac{v_p}{f} \ [m] \qquad (2.27)$$

3. The Wave Impedance or medium intrinsic impedance, η

- In section 2.3.2, one of wave equations is solved [(2.16) for electric field $\vec{E}$ or (2.17) for the magnetic field $\vec{H}$.
- In general, whenever $\vec{E}$ or $\vec{H}$ is known, the other filed vector can be readily found by using one of Maxwell's curl equations.
- Thus applying (2.10) to the electric field obtained (2.18) gives: $\nabla x \vec{E}_s = -j\omega\mu\vec{H}_s$

$$Curl\vec{E}_s = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & E_{ys} & E_{zs} \end{vmatrix} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs}(z) & 0 & 0 \end{vmatrix} = 0\hat{a}_x - \frac{\partial E_{xs}(z)}{\partial z}\hat{a}_y + 0\hat{a}_z = -j\omega\mu\vec{H}_s$$

2.3 Solution of Wave Equations in a General Medium (continued)

2.3.3 Basic plane wave parameters (continued)

3. The Wave Impedance or medium intrinsic impedance, η

- Which is greatly simplified for a single E_x component varying only with z and $H_x = H_z = 0$, we get: $$\frac{dE_{xs}(z)}{dz} = -j\omega\mu H_{ys}$$
- Substitute using Eq.(2.21) for E_{sx}, we have: $E_{xs}(z) = E^+ e^{-\gamma z} = E^+ e^{-\alpha z} e^{-j\beta z}$

$$H_{ys} = -\frac{1}{j\omega\mu}(-\gamma)E^+ e^{-\gamma z} = \frac{\gamma}{j\omega\mu} E^+ e^{-\alpha z} e^{-j\beta z} \qquad (2.28a)$$

- In real instantaneous form, (2.28a) becomes:

$$H_y(z,t) = \frac{\gamma}{j\omega\mu} E^+ e^{-\alpha z}\cos(\omega t - \beta z) \Rightarrow E_x(z,t) = \eta\, H_y(z,t) \qquad (2.28b)$$

- Generally, The magnetic field intensity, H is related to electric field intensity, E by:

$$\vec{E} = \eta\vec{H} \quad \Rightarrow \vec{H} = \frac{\vec{E}}{\eta} \quad \eta = \frac{|\vec{E}|}{|\vec{H}|} = \frac{in\ [V/m]}{in\ [A/m]} \Rightarrow in\ [\Omega] \qquad (2.28c)$$

- Where η is called ***wave impedance*** for the plane wave or the intrinsic impedance for the general medium is given by:

$$\eta = \frac{j\omega\mu}{\gamma}\ [\Omega] \qquad (2.29)$$

2.3 Solution of Wave Equations in a General Medium (continued)

2.3.3 Basic plane wave parameters (continued)

4. Skin Depth

$$\alpha \ in \ [Np/m]$$

- From Eq. (2.22), it is seen that as $\vec{E}$ and $\vec{H}$ waves travel in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ, through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called skin depth or penetration depth of the medium as shown in Fig. 2.3.

$$E^+e^{-\alpha\delta} = E^+e^{-1} = 0.368E^+$$

$$\alpha\delta_s = 1 \ \Rightarrow \delta_s = \frac{1}{\alpha} \ [m] \ (2.30)$$

The skin depth is a measure of the depth to which an electromagnetic wave can penetrate the medium.

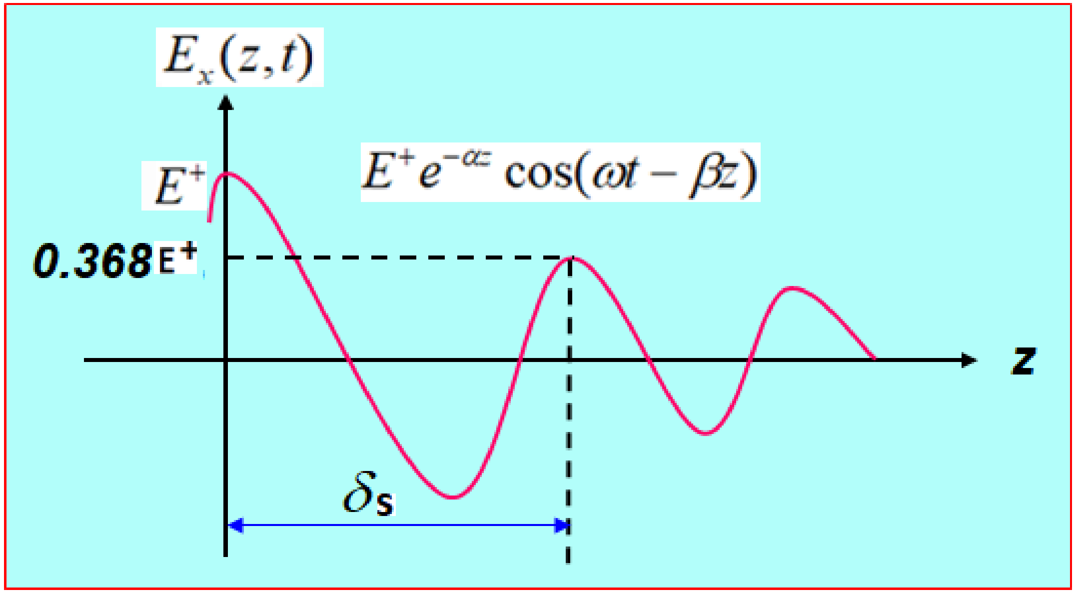


Fig. 2.3 Skin depth

2.3 Solution of Wave Equations in a General Medium (continued)

Example 2.1: A uniform plane wave is propagating + z direction in in a medium (ε_r = 72, μ_r = 1, σ = 4 (S/m)), If the operating frequency, f = 1 GHz and its electric field intensity at z = 0 and t = 0 is $|\vec{E}|$= 10 mV/m and directed toward the x- axis. Determine and calculate the following:

a) The attenuation constant, α and phase constant, β also the wave parameters: phase velocity, v_p, wavelength, λ, intrinsic impedance, η and skin depth δ_s.

b) Write the expressions for the electric and magnetic field intensities $\vec{E}$ and $\vec{H}$.

Solution

a) The angular frequency, ω = 2πf = 2π x 10^9 = 2π G rad/s.

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi x 10^9 x72(1/36\pi)x10^{-9}} = \frac{4x36\pi}{2\pi x72} = 1$$

The complex propagation constant, γ for this medium is given by Eq. (2.16):

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} = j2\pi x10^9\sqrt{1x4\pi x10^{-7}x72x(1/36\pi)x10^{-9}}\sqrt{1 - j1}$$

$$= j2\pi x10^9 x2.83x10^{-8}x\sqrt{\sqrt{2}e^{-j\pi/4}} = j1066.88[1.18\cos(-22.5^o) - j1.18\sin(22.5^o)]$$

$$= 481.77 + j1163.09$$

so $$\alpha = 481.77 \ \ Np/m \ and \ \ \beta = 1163.09$$

2.3 Solution of Wave Equations in a General Medium (continued)

Example 2.1 Solution: The wave parameters are calculated as follows:

From Eq. (2.26), the phase velocity, v_p is given by:

$$v_p = \frac{\omega}{\beta} = \frac{2\pi x 10^9}{1163.09} = 0.054 x 10^8 \ [m/s]$$

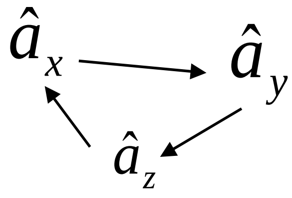
From Eq. (2.27), the wavelength, λ is given by:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1163.09} = 5.4 \ [mm]$$

From Eq. (2.29), the intrinsic impedance, η is given by:

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j2\pi x 10^9 x 4\pi x 10^{-7}}{481.77 + j1163.09} = \frac{j2\pi x 10^9 x 4\pi x 10^{-7}}{481.77 + j1163.09}$$

$$= \frac{j7898.68}{481.77 + j1163.09} = 5.78 + j2.38 = 6.25\angle 22.4^o \ [\Omega]$$



From Eq. (2.30), the skin depth δ_s is given by: $\delta_s = \frac{1}{\alpha} = \frac{1}{481.77} = 2.08 \ [mm]$

b) From Eq. (2.22), The electric field intensity $\vec{E}$ is given by:

$$\vec{E}(z,t) = E^+ e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_x = 10 \ e^{-481.77z}\cos(\omega t - 1163.0z)\hat{a}_x \ mV/m$$

From Eq. (2.28), The magnetic field intensity $\vec{H}$ is given by: :

$$\vec{H}(z,t) = \frac{E_x(z,t)}{\eta}\hat{a}_y = \frac{10 \ e^{-481.77z}\cos(\omega t - 1163.0z)}{6.25\angle 22.4^o}\hat{a}_y$$

$$\theta^{rad} = \frac{\theta^o}{180}\pi$$

$$= 1.6e^{-481.77z}\cos(\omega t - 1163.0z - 0.124\pi)\hat{a}_y \ mA/m$$

2.4 Solution of Wave Equations in Free Space

- When considering electromagnetic waves, in free space, it is called sourceless ($\rho_v = 0$, $J = 0$, $\sigma = 0$, $\mu = \mu_o$ and $\varepsilon = \varepsilon_o$). Under these conditions, the complex propagation constant, γ for the medium defined Eq. (2.16c) is given by:

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} = j\omega\sqrt{\mu_o\varepsilon_o}\sqrt{1 - j\frac{0}{\omega\varepsilon}} = j\omega\sqrt{\mu_o\varepsilon_o} \qquad (2.30a)$$

- Therefore, $\alpha = 0$, $\beta = \omega\sqrt{\mu_o\varepsilon_o} = k_o$ and $\gamma^2 = -\omega^2\mu_o\varepsilon_o$ (2.30b)

- The waves equations (2.16) and (2.17) become:

$$\alpha = 0 \ , \quad \beta = \omega\sqrt{\mu_0\varepsilon_0} = \omega / c$$

$$\nabla^2\vec{E}_s - \gamma^2\vec{E}_s = 0 \quad \Rightarrow \nabla^2\vec{E}_s + \omega^2\mu_o\varepsilon_o\vec{E}_s = 0$$

so $\nabla^2\vec{E}_s + k_o^2\vec{E}_s = 0$ (2.31) and $\nabla^2\vec{H}_s + k_o^2\vec{H}_s = 0$ (2.32)

The constant $k_o = \omega\sqrt{\mu_o\varepsilon_o}$ is called ***the wave number or propagation constant.***

- The solution to Eq.(2.31) will be forward and backward propagating waves having the phasor solutions in the form:

$$E_{xs}(z) = E^+e^{-\gamma z} + E^-e^{\gamma z} = E^+e^{-jk_oz} + E^+e^{jk_oz} \qquad (2.33a)$$

- The above solution is for time harmonic case at frequency ω. In time domain, this result is written as: $E_x(z,t) = \mathrm{Re}[E_{xs}(z)e^{j\omega t}]$

$$E_x(z,t) = E^+\cos(\omega t - k_oz) + E^-\cos(\omega t + k_oz) \qquad (2.33b)$$

2.4 Solution of Wave Equations in Free Space (continued)

- The magnetic field intensity, H is related to electric field intensity, E by:

$$\vec{E} = \eta_o \vec{H} \quad \Rightarrow \vec{H} = \vec{E} / \eta_o \tag{2.34}$$

- Note the following characteristics of the forward wave in eq.(2.33):

$$E_x(z,t) = E^+ \cos(\omega t - k_o z)$$

1. It is time harmonic because we assumed time dependence $e^{j\omega t}$ to arrive .
2. E^+ is called the amplitude of the wave and has the same units as E.
3. (ωt - k_0 z) is the phase (in radians) of the wave; it depends on time t and space variable z.
4. ω is the angular frequency (rad/s) and k_0 is the phase constant or wave number (rad/m).

- Due to the variation of E with both time t and space variable z, we may plot E as a function of t by keeping z constant and vice versa. The plots of E(z, t = constant) and E(t, z = constant) are as in Fig. 2.4(a) and (b).
- From Fig. 2.4(a), we observe that the wave takes distance λ to repeat itself and hence λ is called the wavelength (in meters).
- From Fig. 2.4(b), the wave takes time T to repeat itself; consequently T is known as the period (in seconds). Since it takes time T for the wave to travel
- distance λ at the speed v, we expect that: $\lambda = v_p T = v_p / f$

2.4 Solution of Wave Equations in Free Space (continued)

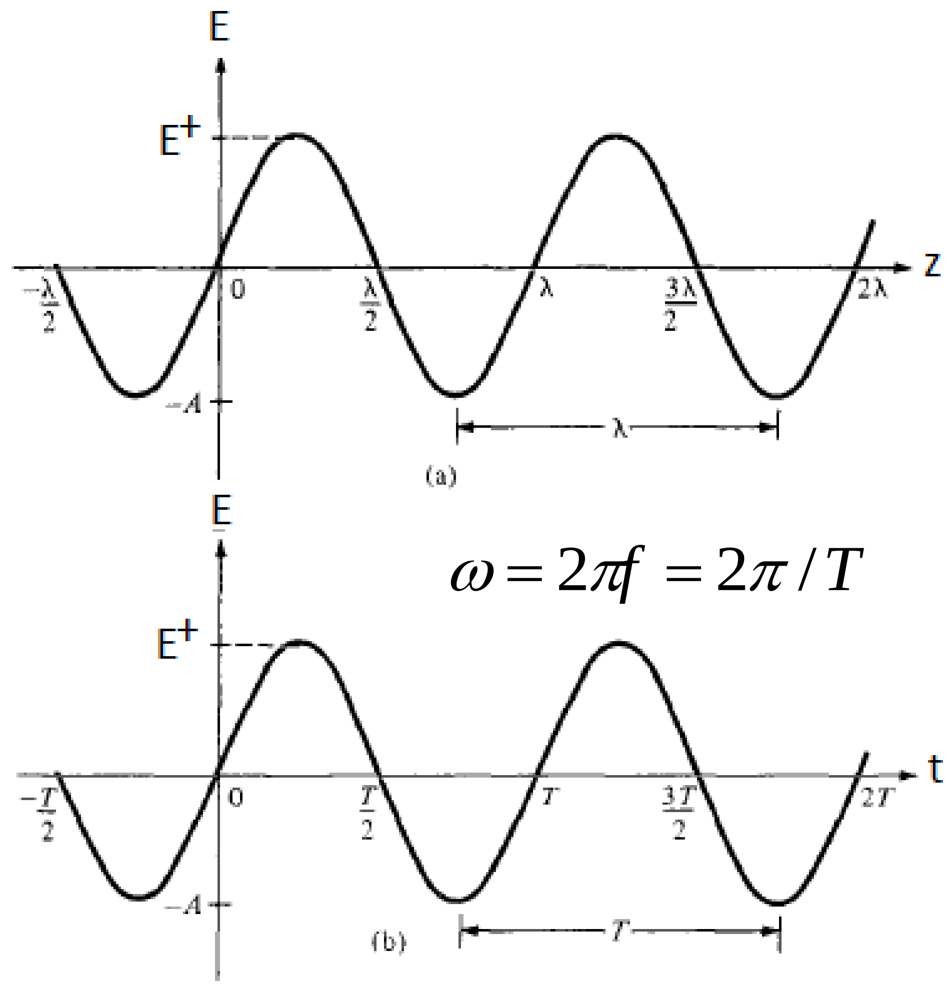
- But T = l/f, where f is the frequency of the wave in (Hz).

$$\lambda = v_p T = \frac{v_p}{f} \Rightarrow v_p = f\lambda$$

- Because of this fixed relationship between wavelength and frequency, one can identify the position of a radio station within its band by either the frequency or the wavelength. Usually the frequency is preferred.

$$k_o = 2\pi / \lambda_o$$

- Which shows that for every wavelength of distance traveled, a wave undergoes a phase change of 2π radians.



$$\omega = 2\pi f = 2\pi / T$$

Fig. 2.4 Plot of $E_x(z,t) = E^+ \sin(\omega t - k_o z)$: (a) with constant t and (b) with constant z.

2.4 Solution of Wave Equations in Free Space (continued)

Basic plane wave parameters

1. Phase velocity, v_p

- The phase velocity, v_p in the general medium is given by Eq. (2.26) as:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} \qquad \omega \text{ in } [rad/s] \text{ and } k_o \text{ in } [rad/m] \qquad (2.26)$$

since $\beta = k_o = \omega\sqrt{\mu_o \varepsilon_o}$ $\quad v_p = \frac{\omega}{k_o} = \frac{\omega}{\omega\sqrt{\mu_o\varepsilon_o}} = \frac{1}{\sqrt{\mu_o\varepsilon_o}} \; [m/s]$ (2.35)

- For free space, the phase velocity is given by:

$$v_p = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = \frac{1}{\sqrt{4\pi x 10^{-7} x \frac{1}{36\pi} x 10^{-9}}} = 3x10^8 m/\sec = c$$

c is the speed of light in the free space.

2. The Wavelength, λ

- The wavelength, λ in the general medium is given by Eq. (2.27) as: $\lambda = \frac{2\pi}{\beta}\; [m]$

So, for free space, λ_o is given by:

$$\lambda_o = \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{k_o} = \frac{2\pi}{\omega\sqrt{\mu_o\varepsilon_o}} = \frac{2\pi}{2\pi f\sqrt{\mu_o\varepsilon_o}} = \frac{c}{f} \; [m] \qquad (2.36)$$

2.4 Solution of Wave Equations in Free Space (continued)

Basic plane wave parameters

3. The Wave Impedance or medium intrinsic impedance, η

- The intrinsic impedance for the general medium is given by Eq. (2.29):

$$\eta = \frac{j\omega\mu}{\gamma} \ [\Omega]$$

So, for free space, the intrinsic impedance, η_o is given by:

$$\eta_o = \eta = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{j\beta} = \frac{\omega\mu_o}{k_o} = \frac{\omega\mu_o}{\omega\sqrt{\mu_o\varepsilon_o}} = \sqrt{\frac{\mu_o}{\varepsilon_o}} \ [\Omega] \qquad (2.37)$$

$$\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi x 10^{-7}}{10^{-9}/36\pi}} = \sqrt{144\pi^2 x 10^2} = 120\pi \ \Omega = 377 \ \Omega$$

4. Skin Depth

- The skin depth for the general medium is given by Eq. (2.30): $\delta_s = \frac{1}{\alpha} \ [m]$

So, for free space:

$$\delta_s = \frac{1}{\alpha} = \frac{1}{0} = \infty \qquad (2.38)$$

- Therefore, the electromagnetic wave can propagate without attenuation in the free space.

2.4 Solution of Wave Equations in Free Space (continued)

Basic plane wave parameters (continued)

Example 2.2

A uniform plane wave in free space its electric field intensity, $\vec{E}$ is given by:

$\vec{E} = 0.8\cos(2\pi x10^8 t - k_o y)\hat{a}_z \;\; mV/m$. Find:

a) The propagation constant, k_o. b) The wavelength, λ_o.

c) The magnetic field intensity, $|\vec{H}|$ at the point $P(0.1, 1.5, 0.4)$ at t = 8 ns.

Solution a) The propagation constant, k_o is given by:

$$k_o = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} = \frac{2\pi x10^8}{3x10^8} = 2.094 \;\; rad/m$$

b) The wavelength, λ_o is given by: $\lambda_o = \dfrac{c}{f} = \dfrac{3x10^8}{10^8} = 3 \;\; m$

c) The magnetic field intensity, $\vec{H}$ is given by (2.28) as:

$$\vec{H} = \frac{0.8x10^{-3}\cos(2\pi x10^8 t - k_o y)}{\eta_o}\hat{a}_x = \frac{0.8x10^{-3}\cos(2\pi x10^8 t - 2.094y)}{120\pi}\hat{a}_x$$

$$= 2.12\cos(2\pi x10^8 t - 2.094y)\hat{a}_x \;\; \mu A/m$$

At the point $P(0.1, 1.5, 0.4)$ at t = 8 ns, we have:

$$|\vec{H}| = 2.12\cos(2\pi x10^8 x8x10^{-9} - 2.094x1.5)\hat{a}_x = 12.2|\cos 108.03^o| = 3.68 \;\; \mu A/m$$

2.4 Solution of Wave Equations in Free Space (continued)

Basic plane wave parameters (continued)

Example 2.3: In the free space if the magnetic field intensity, $\vec{H}$ of a plane wave is given by: $\vec{H}(z,t) = 2.65\sin(\omega t - \beta\ z)\hat{a}_y\ mA/m$

If the working frequency is f = 10 GHz, determine and calculate the following:

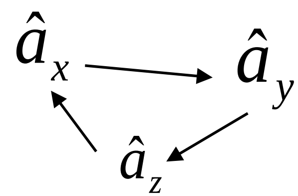
a) The basic plane wave parameters, v_p , λ_o and η_o.

b) The electric field intensity, $\vec{E}$.

Solution a) The propagation constant, k_o is given by:

$$k_o = \beta = \omega\sqrt{\mu_o \varepsilon_o} = 2\pi x 10^{10}\sqrt{4\pi x 10^{-7}\, x(1/36\pi)10^{-9}} = \frac{\omega}{c} = \frac{2\pi x 10^{10}}{3x10^8} = 209.4\ \ rad/m$$

The phase velocity, v_p in the general medium is given by:



$$\upsilon_p = c = \frac{\omega}{\beta} = \frac{2\pi x 10^9}{209.4} = 3x10^8\ \ m/\sec$$

The wavelength, λ_o is given by: $\lambda_o = \frac{2\pi}{\beta} = \frac{2\pi}{k_o} = \frac{2\pi}{209.4} = \frac{c}{f} = 0.03\ \ m = 3\ \ cm$

The intrinsic impedance, η_o is:

$$\eta = \eta_o = \frac{\omega\mu_o}{\beta} = \frac{2\pi x 10^{10}\, x 4\pi x 10^{-7}}{209.4} = 377\ \ \Omega = 120\pi\ \ \Omega$$

b) The electric field intensity, $\vec{E}$ is given by (2.28) as:

$$\vec{E}(z,t) = 2.65 x \eta_o \sin(\omega t - \beta\ z)\hat{a}_x\ \ = 1.0\sin(2\pi x 10^{10} t - 20.94\ \ z)\hat{a}_x\ V/m$$

2.5 Solution of Wave Equations in a Lossless Medium

- When considering electromagnetic waves, in lossless material where μ and ε real ($\rho_v = 0$, J = 0, σ = 0 or σ << ωε, $\mu = \mu_r\mu_o$ and $\varepsilon = \varepsilon_r\varepsilon_o$). Under these conditions, the complex propagation constant, γ for the medium defined Eq. (2.16c) is given by:

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{0}{\omega\varepsilon}} = j\omega\sqrt{\mu\varepsilon} \qquad (2.39a)$$

- Therefore, $\alpha = 0$, $\beta = \omega\sqrt{\mu\varepsilon} = k$ and $\gamma^2 = -\omega^2\mu\varepsilon$ (2.39b)

- The waves equations (2.16) and (2.17) become:

$$\nabla^2\vec{E}_s - \gamma^2\vec{E}_s = 0 \quad \Rightarrow \nabla^2\vec{E}_s + \omega^2\mu\varepsilon\vec{E}_s = 0$$

so $\nabla^2\vec{E}_s + k^2\vec{E}_s = 0$ (2.40) and $\nabla^2\vec{H}_s + k^2\vec{H}_s = 0$ (2.41)

The constant $k = \omega\sqrt{\mu\varepsilon}$ is called ***the wave number or propagation constant.***

- The solution to Eq.(2.40) will be forward and backward propagating waves having the phasor solutions in the form:

$$E_{xs}(z) = E^+e^{-\gamma z} + E^-e^{\gamma z} = E^+e^{-jkz} + E^-e^{jkz} \qquad (2.42a)$$

- The above solution is for time harmonic case at frequency ω. In time domain, this result is written as: $E_x(z,t) = \text{Re}[E_{xs}(z)e^{j\omega t}]$

$$E_x(z,t) = E^+\cos(\omega t - kz) + E^-\cos(\omega t + kz) \qquad (2.42b)$$

2.5 Solution of Wave Equations in a Lossless Medium (continued)

Basic plane wave parameters

- The magnetic field intensity, H is related to electric field intensity, E by:

$$\vec{E} = \eta\vec{H} \quad \Rightarrow \vec{H} = \vec{E}/\eta \qquad (2.43)$$

1. Phase velocity, v_p

- The phase velocity, v_p in the a lossless medium is given by Eq. (2.26) as: $v_p = \dfrac{\omega}{\beta}$

since $\beta = k = \omega\sqrt{\mu\varepsilon}$

$$v_p = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_r\mu_o\varepsilon_r\varepsilon_o}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} \quad [m/s] \qquad (2.44)$$

2. The Wavelength, λ

- The wavelength, λ in a lossless medium is given by Eq. (2.27) as: $\lambda = \dfrac{2\pi}{\beta}$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{2\pi}{2\pi f\sqrt{\mu_r\mu_o\varepsilon_r\varepsilon_o}} = \frac{c}{f\sqrt{\mu_r\varepsilon_r}} = \frac{v_p}{f} \quad [m] \qquad (2.45)$$

3. The Wave Impedance or medium intrinsic impedance, η

- The intrinsic impedance for a lossless medium is given by Eq. (2.29): $\eta = \dfrac{j\omega\mu}{\gamma}$

$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r\mu_o}{\varepsilon_r\varepsilon_o}} = \eta_o\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} \quad [\Omega] \qquad (2.46)$$

4. Skin Depth

- The skin depth for the general medium is given by Eq. (2.30): $\delta_s = \dfrac{1}{\alpha} \; [m]$

$$\delta = 1/\alpha = 1/0 = \infty \qquad (2.47)$$

The electromagnetic wave can propagate without attenuation a lossless medium

2.5 Solution of Wave Equations in a Lossless Medium (continued)

Basic plane wave parameters (continued)

Example 2.4: A 9.4 GHz uniform plane wave is propagating in polyethylene (ε_r=2.26, μ_r=1) in the + z direction. If the amplitude of the magnetic field intensity, $\vec{H}$ is 7 mA/m, directed in the y axes and the material is assumed to be lossless, determine and calculate the following:

a) The basic plane wave parameters, v_p , λ and η.

b) The magnetic and electric field intensities, $\vec{H}$ and $\vec{E}$.

Solution a) The propagation constant, k is given by:

$$k = \beta = \omega\sqrt{\mu\varepsilon} = 2\pi x 9.4 x 10^9 \sqrt{4\pi x 10^{-7} x 2.26 x (1/36\pi) 10^{-9}} = 295.965 \ rad/m$$

The phase velocity, v_p in the general medium is given by:

$$\upsilon_p = \frac{\omega}{\beta} = \frac{2\pi x 9.4 x 10^9}{295.965} = 1.9956 x 10^8 \ m/\sec$$

The wavelength, λ is given by: $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{k} = \dfrac{2\pi}{295.965} = 0.021 \ m = 2.1 \ cm$

The intrinsic impedance, η is:

$$\eta = \frac{\omega\mu}{\beta} = \frac{2\pi x 9.4 x 10^9 x 4\pi x 10^{-7}}{295.965} = 251.37 \ \Omega$$

b) The magnetic and electric field intensities, $\vec{H}$ and $\vec{E}$ are given by:

$$\vec{H}(z,t) = 7\cos(\omega t - \beta \ z)\hat{a}_y = 7\cos(2\pi x 9.4 x 10^9 t - 295.965 \ z)\hat{a}_y \ mA/m$$

$$\vec{E}(z,t) = 7 x \eta \cos(\omega t - \beta \ z)\hat{a}_x = 1.76\cos(2\pi x 9.4 x 10^9 t - 295.695 \ z)\hat{a}_x \ V/m$$

2.5 Solution of Wave Equations in a Lossless Medium (continued)

Example 2.5 A plane wave propagating in lossless dielectric medium ($\varepsilon_r > 1$, $\mu_r = 1$ and $\sigma = 0$) has an electric field given as $E_x = E^+ \cos(1.51x\ 10^{10} t - 61.6\ z)$. For this wave, determine the:

(a) Wavelength, λ.

(b) Phase velocity, v_p.

(c) Dielectric constant of the medium, ε_r.

(d) Intrinsic or Wave impedance, η

Solution

(a) By comparison with (2.19) we identify $\omega = 1.51 \times 10^{10}$ rad /sec and k= 61.6 m^{-1}

$\gamma = \alpha + j\beta = j\ 61.6\ m^{-1}$, then $\alpha = 0$ and $k = \beta = 61.6$ which gives the wavelength λ as:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{k} = \frac{2\pi}{61.6} = 0.102m$$

(b) The phase velocity can be found from:

$$\upsilon_p = \frac{\omega}{\beta} = \frac{\omega}{k} = \frac{1.51x10^{10}}{61.6} = 2.45x10^8\, m/\sec$$

(c) This is slower than the speed of light by a factor of 1.225. The dielectric constant, ε_r of the medium can be found as:

$$v_p = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{c}{\sqrt{\varepsilon_r}} \quad gives \quad \in_r = (\frac{c}{\upsilon_p})^2 = (\frac{3.0x10^8}{2.45x10^8})^2 = 1.50$$

(c) The intrinsic or wave impedance, η is given by:

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{k} = \frac{\eta_o}{\sqrt{\in_r}} = \frac{377}{\sqrt{1.5}} = 307.8\ \Omega$$

2.6 Solution of Wave Equations in a Good Conductor

- When considering electromagnetic waves, in conducting medium ($\rho_v = 0$, $\sigma \neq 0$, $\mu = \mu_r\mu_o$ and $\varepsilon = \varepsilon_r\varepsilon_o$ also $\sigma >> \omega\varepsilon$). Under these conditions, The waves equations (2.16) and (2.17) are:

$$\nabla^2\vec{E}_s - \gamma^2\vec{E}_s = 0 \quad , \quad \nabla^2\vec{H}_s - \gamma^2\vec{H}_s = 0 \text{ and } \gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

- Since $\sigma \gg \omega\varepsilon$ metals can be categorized as good conductors, so the one can be neglected with respect to $\sigma/\omega\varepsilon$ ($1 << \sigma/\omega\varepsilon$) and the complex propagation constant γ is given by:

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{-j\frac{\sigma}{\omega\varepsilon}} = j\sqrt{-j\omega\mu\sigma}$$

but $\quad -j = 1\angle -90 \quad and \quad \sqrt{1\angle -90^o} = 1\angle -45^o$

$$\sqrt{-j} = 1\angle -45^o = (\cos 45^0 - j\sin 45^o) = \frac{1}{\sqrt{2}}(1 - j)$$

therefore

$$\gamma = j\sqrt{-j\omega\mu\sigma} = j\frac{(1-j)}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$= (1+j)\sqrt{\frac{2\pi f\mu\sigma}{2}} = (1+j)\sqrt{\pi f\mu\sigma} \qquad (2.48a)$$

2.6 Solution of Wave Equations in a Good Conductor (continued)

Therefore: $\alpha = \beta = \sqrt{\pi f \mu \sigma}$ $\qquad E_{xs}(z) = E^{+}e^{-\gamma z} + E^{-}e^{\gamma z}$ (2.48b)

- Regardless of the parameter μ and σ of the conductor or the frequency of applied field, α and β are equal. If we again assume only an E_x component traveling in the +z direction the Eq.(2.22) becomes: $E_x(z,t) = E^{+}e^{-\alpha z}\cos(\omega t - \beta z)$

$$E_x(z,t) = E^{+}e^{-\alpha z}\cos(\omega t - \beta z) = E^{+}e^{-z\sqrt{\pi f \mu \sigma}}\cos(\omega t - z\sqrt{\pi f \mu \sigma}) \quad (2.49)$$

Definition: The depth of penetration (δ_s) is the distance traveled inside the medium at which the amplitude of the wave attenuates e (e ≈2.7) times from its initial level as shown in Fig. 2.7. This is also called skin depth or skin effect.

- From Eq.(2.44) the field amplitude is given by:

$$E_x(z) = E^{+}e^{-\alpha z} = E^{+}e^{-\sqrt{\pi f \mu \sigma}\, z} \quad (2.50)$$

At z = 0 $|E_x(z)| = E^{+}$ and

at z = δ_s $|Ex(z)| = E^{+}/e$, therefore:

$$|E_x(z)| = E^{+}e^{-\delta_s\sqrt{\pi f \mu \sigma}} = E^{+}e^{-1}$$

- The skin depth or depth of penetration δ_s is:

$$\delta_s = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} = \sqrt{\frac{2}{\omega \mu \sigma}}$$

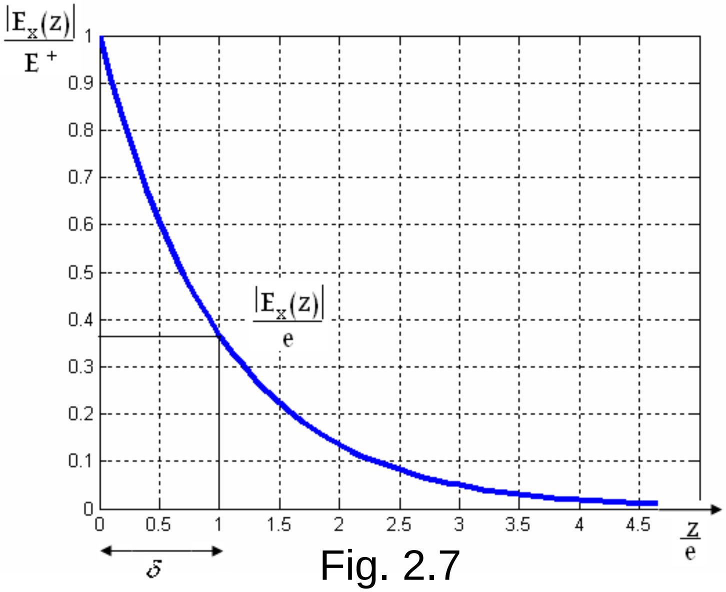


Fig. 2.7

2.6 Solution of Wave Equations in a Good Conductor (continued)

Basic plane wave parameters

- The magnetic field intensity, H is related to electric field intensity, E by:

$$\vec{E} = \eta \vec{H} \quad \Rightarrow \vec{H} = \vec{E}/\eta \qquad (2.51)$$

1. Phase velocity, v_p

- The phase velocity, v_p in a conductor medium is given by Eq. (2.26) as: $v_p = \dfrac{\omega}{\beta}$

since $\alpha = \beta = \sqrt{\pi f \mu \sigma}$

$$v_p = \frac{\omega}{\sqrt{\pi f \mu \sigma}} = \sqrt{\frac{\omega^2}{\pi f \mu \sigma}} = \sqrt{\frac{2\omega}{\mu \sigma}} \ [m/s] \qquad (2.52)$$

2. The Wavelength, λ

- The wavelength, λ in a lossless medium is given by Eq. (2.27) as: $\lambda = \dfrac{2\pi}{\beta}$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\pi f \mu \sigma}} \ [m] \qquad (2.53)$$

3. The Wave Impedance or medium intrinsic impedance, η

- The intrinsic impedance for a lossless medium is given by Eq. (2.29): $\eta = \dfrac{j\omega\mu}{\gamma}$

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{\pi f \mu \sigma} + j\sqrt{\pi f \mu \sigma}} \ [\Omega] \qquad (2.54)$$

4. Skin Depth

- The skin depth for the general medium is given by Eq. (2.30): $\delta_s = \dfrac{1}{\alpha} \ [m]$

$$\delta_s = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} \ [m] \qquad (2.47)$$

The electromagnetic wave can propagate without attenuation a lossless medium

2.6 Solution of Wave Equations in a Good Conductor (continued)

Example 2.6 Compute the skin depth (δ_s) at a frequency of 10 GHz of the following materials with μ_r =1 and conductivity, σ as indicated:

a) Aluminum (σ_{Al} = 3.816x10^7 S/m). b) Gold (σ_{Gl} = 4.098x10^7 S/m).

c) Copper (σ_{Co} = 5.818x10^7 S/m). d) Silver (σ_{Sl} = 6.173x10^7 S/m).

Solution: Eq. (2.46) gives the skin depth δ_s as:

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu_r \mu_o \sigma}} = \sqrt{\frac{1}{\pi(10^{10})(4\pi x 10^{-7})}}\sqrt{\frac{1}{\sigma}} = 5.03x10^{-3}\sqrt{\frac{1}{\sigma}}$$

a) For Aluminum: $$\delta_{s_{Al}} = 5.03x10^{-3}\sqrt{\frac{1}{3.816x10^7}} = 8.14x10^{-7}\,m$$

b) For Gold: $$\delta_{s_{Gl}} = 5.03x10^{-3}\sqrt{\frac{1}{4.098x10^7}} = 7.86x10^{-7}\,m$$

c) For Copper: $$\delta_{s_{Co}} = 5.03x10^{-3}\sqrt{\frac{1}{5.813x10^7}} = 6.60x10^{-7}\,m$$

d) For Silver: $$\delta_{s_{Sl}} = 5.03x10^{-3}\sqrt{\frac{1}{6.173x10^7}} = 6.40x10^{-7}\,m$$

2.6 Solution of Wave Equations in a Good Conductor (continued)

Example 2.7 A plane wave propagating in the sea water medium (ε_r=81, μ_r = 1 and σ = 4 S/m) Consider an incident wave of frequency 1 MHz, find the:

a) Skin depth, δ_s.

b) Wavelength, λ.

c) Phase velocity, v_p .

d) Intrinsic or Wave impedance, η.

Solution

(a) The loss tangent is given by:

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{(2\pi x 10^6)(81)(8.85x10^{-12})} = 890 >> 1$$

Sea water is therefore a good conductor at 1 MHz, the skin, δ_s of Eq. (2.46) is:

$$\delta_s = \frac{1}{\alpha} = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu_o \sigma}} = \sqrt{\frac{1}{\pi(10^6)(4\pi x 10^{-7})x(4)}} = 0.25\ m = 25\ cm$$

(b) the wavelength λ as:

$$\lambda = \frac{2\pi}{\beta} = 2\pi\delta_s = 1.6\ m$$

in free space the wavelength λ_0 = 300 m

(c) The phase velocity can be found from (2.35):

$$\upsilon_p = \frac{\omega}{\beta} = \omega\delta = 2\pi x 10^6 (0.25) = 1.6x10^6\ m/\sec$$

in free space v = c

(d) The intrinsic or wave impedance, η is given by (2.49): $\gamma = \alpha + j\beta = 4 + j4$

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{2\pi x 1x10^6 x 4\pi x 10^{-7} j}{(4 + j4)} = (0.987 + j0.987)\ \Omega = 1.4\angle 45^o\ \Omega$$

2.6 Solution of Wave Equations in a Good Conductor (continued)

Example 2.8 A plane wave is given by: $\vec{E}(z,t) = 0.5\ e^{-4z}\cos(10^9 t - 4z)\,\hat{a}_x\ V/m$

Determine the following:

a) The propagation constant and the wave parameters (V_p, λ, η and δ_s)

b) The magnetic field, associated with the wave in both phasor and time domain representations.

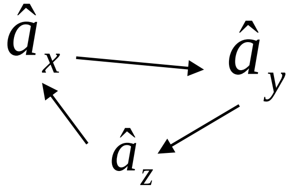
Solution

a) $\vec{E}(z,t) = 0.5\ e^{-4z}\cos(10^9 t - 4z)\,\hat{a}_x\ V/m \qquad \alpha = \beta = 4 \quad Then \quad \gamma = (4 + j4)\ rad/m$

$$\upsilon_p = \frac{\omega}{\beta} = \frac{10^9}{4} = 2.5x10^8\ \ m/\sec \qquad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{4} = \frac{\pi}{2}\ \ m = 1.57m$$

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j10^9 x4\pi x10^{-7}}{(4+j4)} = \frac{10^9 x4\pi x10^{-7}}{(4-j4)} = (50\pi + j50\pi)\,\Omega = 222e^{j\frac{\pi}{4}}\ \Omega$$

$$\delta_s = \frac{1}{\alpha} = \frac{1}{4} = 0.25\ \ m$$



b) $\vec{E}_s(z) = E^+ e^{-\alpha z} e^{-j\beta z}\ \hat{a}_x \qquad \vec{E}_s(z) = 0.5e^{-4z}e^{-j4z}\ \hat{a}_x\ V/m$

$$\vec{H}_s(z) = \frac{E^+}{\eta} e^{-\alpha z} e^{-j\beta z}\ \hat{a}_y \qquad \vec{H}_s(z) = \frac{0.5}{222\ e^{j\frac{\pi}{4}}} e^{-\alpha z} e^{-j\beta z}\ \hat{a}_y = 2.25\ e^{-4z}\ e^{-j(4z+\frac{\pi}{4})}\ \hat{a}_y\ \text{mA/m}$$

$$\vec{H}(z,t) = 2.25\ \ e^{-4z}\cos(10^9 t - 4z - \frac{\pi}{4})\ \ \hat{a}_y\ \ \text{mA}/m$$